The constrained modified KP hierarchy and the generalized Miura transformations

1997 J. Phys. A: Math. Gen. 30 L725

(http://iopscience.iop.org/0305-4470/30/21/004)

J. Phys. A: Math. Gen. **30** (1997) L725–L731. Printed in the UK PII: S0305-4470(97)86347-0

LETTER TO THE EDITOR

The constrained modified KP hierarchy and the generalized Miura transformations

Jiin-Chang Shaw†§ and Ming-Hsien Tu‡‖

† Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, Republic of China

‡ Department of Physics, National Tsing Hua University, Hsinchu, Taiwan, Republic of China

Received 28 July 1997

Abstract. In this letter, we consider the second Hamiltonian structure of the constrained modified KP hierarchy. After mapping the Lax operator to a pure differential operator the second structure becomes the sum of the second and the third Gelfand–Dickey brackets defined by this differential operator. We simplify this Hamiltonian structure by factorizing the Lax operator into linear terms.

Classical W-algebras have played an important role in integrable systems [1]. It's Adler map (see, for example, [2]) from which the W-algebras can be constructed as Poisson bracket algebras. A typical example is the W_n algebra constructed from the second Gelfand–Dickey (GD) structure of the nth Korteweg–de Vries (KdV) hierarchy [3, 4]. Amazingly, under factorization of the KdV–Lax operator, the second Hamiltonian structure is transformed into a much simpler one in an appropriate space of the modified variables. Thus the factorization not only provides a Miura transformation which maps the nth KdV hierarchy to the corresponding modified hierarchies but also gives a free-field realization of the W_n algebra. This is what we called the Kupershmidt–Wilson (KW) theorem [5, 6]. In general, the above scheme is encoded in the particular form of the Lax operator and its associated Poisson structure. Several integrable systems have been studied based on this scheme, such as the Kadomtsev–Petviashvili (KP) hierarchy and its reductions [7–13].

In this letter, we will consider a kind of reduction of the KP hierarchy called the constrained modified KP (cmKP) hierarchy [14]. Many properties of the cmKP have been studied, such as bi-Hamiltonian structure [14], Bäcklund transformation [15], modification [16], and the conformal property [17], etc. However, a clear and conclusive statement about the associated Poisson structure is still lacking. In the following, we will concentrate on this problem. Especially, we will show that there is an interesting property of the second Poisson structure of the cmKP hierarchy under factorization of the Lax operator into linear terms.

The cmKP hierarchy [14] has the Lax operator of the form

$$K_n = \partial^n + v_1\partial^{n-1} + \cdots + v_n + \partial^{-1}v_{n+1} \tag{1}$$

§ E-mail address: shaw@math.nctu.edu.tw

‖ E-mail address: mhtu@phys.nthu.edu.tw

which satisfies the hierarchy equations

$$\partial_k K_n = [(K_n^{k/n})_{\geqslant 1}, K_n]. \tag{2}$$

The second Poisson bracket associated with the Lax operator was obtained by Oevel and Strampp [14] as follows

$$\{F, G\} = \int \operatorname{res}\left(\frac{\delta F}{\delta K_n}\Theta_2\left(\frac{\delta G}{\delta K_n}\right)\right) \tag{3}$$

where F and G are functionals of K_n and

$$\begin{aligned}\Theta_2\left(\frac{\delta G}{\delta K_n}\right) = {} & \left(K_n\frac{\delta G}{\delta K_n}\right)_+ K_n - K_n\left(\frac{\delta G}{\delta K_n}K_n\right)_+ + \left[K_n, \left(K_n\frac{\delta G}{\delta K_n}\right)_0\right] \\ & +\partial^{-1}\operatorname{res}\left[K_n, \frac{\delta G}{\delta K_n}\right]K_n + \left[K_n, \int^x\left(\operatorname{res}\left[K_n, \frac{\delta G}{\delta K_n}\right]\right)\right]\end{aligned} \tag{4}$$

with

$$\frac{\delta G}{\delta K_n} \equiv \frac{\delta G}{\delta v_{n+1}} + \partial^{-1}\frac{\delta G}{\delta v_n} + \cdots + \partial^{-n}\frac{\delta G}{\delta v_1}. \tag{5}$$

Recently, Liu [16] conjectured that if the Lax operator K_n is factorized as

$$K_n = \partial^{-1}(\partial - w_1)\cdots(\partial - w_{n+1}) \tag{6}$$

then in terms of $\{w_i\}$ the Poisson structure (3) can be simplified to

$$\{w_i(x), w_j(y)\} = (1 - \delta_{ij})\delta'(x - y). \tag{7}$$

where $\delta'(x-y) \equiv \partial_x\delta(x-y)$. The cases for $n = 1$ and 2 have been explicitly demonstrated in [16]. However, to the best of our knowledge, a general proof for all n is still lacking. It is the main purpose of this letter to give an elegant and simple proof for the general case.

To simplify the Hamiltonian structure (4) let us consider the operator

$$\begin{aligned}L_{n+1} \equiv \partial K_n &= \partial^{n+1} + v_1\partial^n + (v_2 + v_1')\partial^{n-1} + \cdots + (v_{n+1} + v_n') \\ &\equiv \partial^{n+1} + u_1\partial^n + u_2\partial^{n-1} + \cdots + u_{n+1}\end{aligned} \tag{8}$$

which is a pure differential operator and the variables $\{v_i\}$ and $\{u_i\}$ are related by

$$\begin{aligned} & v_1 = u_1 \\ & v_2 = u_2 - u_1' \\ & \vdots \\ & v_{n+1} = u_{n+1} - u_n' + \cdots (-1)^n u_1^{(n)}. \end{aligned} \tag{9}$$

Proposition 1. With respect to the pure differential operator L_{n+1}, the second Poisson bracket (3) now becomes

$$\{F, G\} = \int \operatorname{res}\left(\frac{\delta F}{\delta L_{n+1}}\Omega\left(\frac{\delta G}{\delta L_{n+1}}\right)\right) \tag{10}$$

where

$$\begin{aligned}\Omega\left(\frac{\delta G}{\delta L_{n+1}}\right) = {} & \left(L_{n+1}\frac{\delta G}{\delta L_{n+1}}\right)_+ L_{n+1} - L_{n+1}\left(\frac{\delta G}{\delta L_{n+1}}L_{n+1}\right)_+ \\ & + \left[L_{n+1}, \int^x\left(\operatorname{res}\left[L_{n+1}, \frac{\delta G}{\delta L_{n+1}}\right]\right)\right].\end{aligned} \tag{11}$$

with

$$\frac{\delta G}{\delta L_{n+1}} \equiv \partial^{-1}\frac{\delta G}{\delta u_{n+1}} + \partial^{-2}\frac{\delta G}{\delta u_n} + \cdots + \partial^{-n-1}\frac{\delta G}{\delta u_1}. \tag{12}$$

Proof. Since

$$\delta F = \int \operatorname{res}\left(\frac{\delta F}{\delta K_n}\delta K_n\right) = \int \operatorname{res}\left(\frac{\delta F}{\delta L_{n+1}}\delta L_{n+1}\right) \tag{13}$$

we have

$$\frac{\delta F}{\delta K_n} = \frac{\delta F}{\delta L_{n+1}}\partial + \mathrm{O}(\partial^{-n-1}) \tag{14}$$

where $\mathrm{O}(\partial^{-n-1})$ denotes terms of order $-n-1$. Then (10) becomes

$$\{F, G\} = \int \operatorname{res}\left(\frac{\delta F}{\delta L_{n+1}}\partial\Theta_2\left(\frac{\delta G}{\delta K_n}\right)\right) \tag{15}$$

here we drop the terms $O(\partial^{-n-1})$ because the order of $\Theta_2(\frac{\delta G}{\delta K_n})$ is $n-1$. Next we want to express $\partial\Theta_2(\frac{\delta G}{\delta K_n})$ in terms of L_{n+1} and $\frac{\delta G}{\delta L_{n+1}}$. By using (8) and (14), we have

$$\begin{aligned}
\left(L_{n+1}\frac{\delta G}{\delta L_{n+1}}\right)_+ &= \left(K_n\frac{\delta G}{\delta K_n}\right)_+ + \left[\partial, \left(K_n\frac{\delta G}{\delta K_n}\right)_{\geqslant 1}\right]\partial^{-1} \\
\left(L_{n+1}\frac{\delta G}{\delta L_{n+1}}\right)_+ L_{n+1} &= \partial\left(K_n\frac{\delta G}{\delta K_n}\right)_+ K_n - \left(K_n\frac{\delta G}{\delta K_n}\right)_0' K_n \\
\operatorname{res}\left[L_{n+1}, \frac{\delta G}{\delta L_{n+1}}\right] &= \operatorname{res}\left[K_n, \frac{\delta G}{\delta K_n}\right] + \left(K_n\frac{\delta G}{\delta K_n}\right)_0'
\end{aligned} \tag{16}$$

which imply that

$$\begin{aligned}
\partial\Theta_2\left(\frac{\delta G}{\delta K_n}\right) &= \left(L_{n+1}\frac{\delta G}{\delta L_{n+1}}\right)_+ L_{n+1} - L_{n+1}\left(\frac{\delta G}{\delta L_{n+1}}L_{n+1}\right)_+ \\
&\quad + \left[L_{n+1}, \int^x \left(\operatorname{res}\left[L_{n+1}, \frac{\delta G}{\delta L_{n+1}}\right]\right)\right] \\
&\equiv \Omega\left(\frac{\delta G}{\delta L_{n+1}}\right).
\end{aligned} \tag{17}$$

□

Besides the standard second GD structure, the last piece of (11) is called the third GD bracket which is compatible with the second one [3]. Hence, under the mapping (8), the Hamiltonian structure (4) has been mapped to the sum of the second and the third GD structure defined by the differential operator L_{n+1}.

Now we want to show that this Hamiltonian structure can be simplified via the following factorization

$$L_{n+1} = (\partial - w_1)(\partial - w_2)\ldots(\partial - w_{n+1}). \tag{18}$$

This yields an expression for each u_i (and hence v_i) as a differential polynomial in $\{w_i\}$ (the inverse statement is not true). For example

$$\begin{aligned}
u_1 &= -(w_1 + \cdots + w_{n+1}) \\
u_2 &= \sum_{i<j} w_i w_j - \sum_{i=0}^{n-1}(n-i)w'_{n+1-i} \\
&\vdots
\end{aligned} \tag{19}$$

etc. Expression (19) is called the Miura transformation.

Proposition 2. Under the factorization (18), the Poisson structure (10) becomes

$$\{F, G\} = \sum_{i \neq j} \int \left(\frac{\delta F}{\delta w_i}\right)\left(\frac{\delta G}{\delta w_j}\right)' \tag{20}$$

i.e. the basic building blocks $\{w_i\}$ satisfy (7).

Proof. First, thanks to the KW theorem [5, 6] for the second GD structure, the first two terms of the Poisson bracket (10) can be simplified as follows

$$\{F, G\}_2^{\mathrm{GD}} = -\sum_{i=1}^{n+1} \int \left(\frac{\delta F}{\delta w_i}\right)\left(\frac{\delta G}{\delta w_i}\right)' \tag{21}$$

or

$$\{w_i(x), w_j(y)\}_2^{\mathrm{GD}} = -\delta_{ij}\delta'(x - y). \tag{22}$$

Thus the remaining tasks are to verify

$$\int \mathrm{res}\left(\frac{\delta F}{\delta L_{n+1}}\left[L_{n+1}, \int^x \mathrm{res}\left[L_{n+1}, \frac{\delta G}{\delta L_{n+1}}\right]\right]\right) = \sum_{i,j=1}^{n+1} \int \left(\frac{\delta F}{\delta w_i}\right)\left(\frac{\delta G}{\delta w_j}\right)'. \tag{23}$$

Let $l_i \equiv (\partial - w_i)$, then $L_{n+1} = l_1 l_2 \cdots l_{n+1}$ and

$$\begin{aligned}\int \mathrm{res}\left(\frac{\delta F}{\delta L_{n+1}}\delta L_{n+1}\right) &= -\int \mathrm{res}\left(\frac{\delta F}{\delta L_{n+1}}\sum_{i=1}^{n+1} l_1 \ldots l_{i-1}\delta w_i l_{i+1} \ldots l_{n+1}\right) \\ &= -\sum_{i=1}^{n+1}\int \mathrm{res}\left(l_{i+1}\ldots l_{n+1}\frac{\delta F}{\delta L_{n+1}} l_1 \ldots l_{i-1}\right)\delta w_i \\ &= \sum_{i=1}^{n+1}\int \frac{\delta F}{\delta w_i}\delta w_i\end{aligned} \tag{24}$$

which implies

$$\frac{\delta F}{\delta w_i} = -\,\mathrm{res}\left(l_{i+1}\ldots l_{n+1}\frac{\delta F}{\delta L_{n+1}} l_1 \ldots l_{i-1}\right). \tag{25}$$

Now

$$\begin{aligned}\left(\sum_{i=1}^{n+1}\frac{\delta F}{\delta w_i}\right)' &= -\left[\partial, \mathrm{res}\left(\sum_{i=1}^{n+1} l_{i+1}\ldots l_{n+1}\frac{\delta F}{\delta L_{n+1}} l_1 \ldots l_{i-1}\right)\right] \\ &= -\sum_{i=1}^{n+1}\mathrm{res}\left(\left[\partial, l_{i+1}\ldots l_{n+1}\frac{\delta F}{\delta L_{n+1}} l_1 \ldots l_{i-1}\right]\right) \\ &= -\sum_{i=1}^{n+1}\mathrm{res}\left(\left[l_i, l_{i+1}\ldots l_{n+1}\frac{\delta F}{\delta L_{n+1}} l_1 \ldots l_{i-1}\right]\right) \\ &= -\,\mathrm{res}\left[L_{n+1}, \frac{\delta F}{\delta L_{n+1}}\right].\end{aligned} \tag{26}$$

Hence,

$$\sum_{i=1}^{n+1}\frac{\delta F}{\delta w_i} = -\int^x \mathrm{res}\left[L_{n+1}, \frac{\delta F}{\delta L_{n+1}}\right]. \tag{27}$$

Note that we have substituted l_i for ∂ in the third line because nothing will change.

Therefore,

$$\sum_{i,j=1}^{n+1}\int\left(\frac{\delta F}{\delta w_i}\right)\left(\frac{\delta G}{\delta w_j}\right)' = -\int\left(\sum_{i=1}^{n+1}\frac{\delta F}{\delta w_i}\right)'\left(\sum_{j=1}^{n+1}\frac{\delta G}{\delta w_j}\right)$$
$$= -\int \mathrm{res}\left(\left[L_{n+1},\frac{\delta F}{\delta L_{n+1}}\right]\int^x \mathrm{res}\left[L_{n+1},\frac{\delta G}{\delta L_{n+1}}\right]\right)$$
$$= \int \mathrm{res}\left(\frac{\delta F}{\delta L_{n+1}}\left[L_{n+1},\int^x \mathrm{res}\left[L_{n+1},\frac{\delta G}{\delta L_{n+1}}\right]\right]\right). \tag{28}$$

□

Proposition 3. If the Hamiltonian H_k of the cmKP hierarchy equations $\partial_k K_n = [(K_n^{k/n})_{\geqslant 1}, K_n] = \Theta_2(\frac{\delta H_k}{\delta K_n})$ with respect to the second structure is expressed in terms of $\{w_i\}$ by the Miura transformation, then the corresponding modified equations will be

$$\partial_k w_i = \sum_{j\neq i}\left(\frac{\delta H_k}{\delta w_j}\right)'. \tag{29}$$

Proof. This is just a corollary of proposition 2. □

Finally, we would like to provide another interesting property of the Poisson structure (10) although it is less relevant to this case. In fact, in [18] it was shown that the Poisson structure (10) can be associated to the Lax operator of the form

$$L = \partial^N + u_1\partial^{N-1} + \ldots + u_N + \sum_{i=1}^{M}\phi_i\partial^{-1}\psi_i. \tag{30}$$

Therefore we can discuss the Poisson structure (10) under the factorization of the Lax operator containing inverse linear terms.

Proposition 4. Let L be a pseudo-differential operator of the form (30). Then under the following factorization (generalized Miura transformation)

$$L = (\partial - a_1)\ldots(\partial - a_n)(\partial - b_1)^{-1}\ldots(\partial - b_m)^{-1} \qquad (n = N + M, m = M) \tag{31}$$

the Poisson structure (10) associated with L becomes

$$\begin{aligned}&\{a_i(x), a_j(y)\} = (1-\delta_{ij})\delta'(x-y)\\&\{b_i(x), b_j(y)\} = (1+\delta_{ij})\delta'(x-y)\\&\{a_i(x), b_j(y)\} = \delta'(x-y).\end{aligned} \tag{32}$$

Proof. It has been shown [10–13] that the second GD bracket with respects to the factorization (31) are given by

$$\begin{aligned}&\{a_i(x), a_j(y)\}_2^{\mathrm{GD}} = -\delta_{ij}\delta'(x-y)\\&\{b_i(x), b_j(y)\}_2^{\mathrm{GD}} = \delta_{ij}\delta'(x-y)\\&\{a_i(x), b_j(y)\}_2^{\mathrm{GD}} = 0.\end{aligned} \tag{33}$$

Hence, we only need to treat the third structure and to show that

$$\int \mathrm{res}\left(\frac{\delta F}{\delta L}\left[L,\int^x \mathrm{res}\left[L,\frac{\delta G}{\delta L}\right]\right]\right) = \int\left(\sum_{i=1}^{n}\frac{\delta F}{\delta a_i} + \sum_{j=1}^{m}\frac{\delta F}{\delta b_j}\right)\left(\sum_{i=1}^{n}\frac{\delta G}{\delta a_i} + \sum_{j=1}^{m}\frac{\delta G}{\delta b_j}\right)'. \tag{34}$$

Let $A_i = (\partial - a_i)$ and $B_j = (\partial - b_j)$ then

$$\delta F = \int \operatorname{res}\left(\frac{\delta F}{\delta L}\delta L\right) = \int \operatorname{res}\left(\frac{\delta F}{\delta L}\sum_{i=1}^{n} A_1 \ldots A_{i-1}\delta A_i \ldots A_n B_1^{-1} \ldots B_m^{-1}\right)$$

$$+\int \operatorname{res}\left(\frac{\delta F}{\delta L} A_1 \ldots A_n \sum_{j=1}^{m} B_1^{-1} \ldots B_{j-1}^{-1}\delta B_j^{-1} \ldots B_m^{-1}\right) \tag{35}$$

$$\equiv \int \left(\sum_{i=1}^{n}\frac{\delta F}{\delta a_i}\delta a_i + \sum_{j=1}^{m}\frac{\delta F}{\delta b_j}\delta b_j\right). \tag{36}$$

Substituting $\delta A_i = -\delta a_i$ and $\delta B_j^{-1} = B_j^{-1}\delta b_j B_j^{-1}$ into (35) and comparing with (36), we obtain

$$\frac{\delta F}{\delta a_i} = -\operatorname{res}\left(A_{i+1} \ldots A_n B_1^{-1} \ldots B_m^{-1}\frac{\delta F}{\delta L} A_1 \ldots A_{i-1}\right) \tag{37}$$

$$\frac{\delta F}{\delta b_j} = \operatorname{res}\left(B_j^{-1} \ldots B_m^{-1}\frac{\delta F}{\delta L} A_1 \ldots A_n B_1^{-1} \ldots B_j^{-1}\right). \tag{38}$$

Thus

$$\sum_{i=1}^{n}\left(\frac{\delta F}{\delta a_i}\right)' + \sum_{j=1}^{m}\left(\frac{\delta F}{\delta b_j}\right)' = \sum_{i=1}^{n}\left[\partial, \frac{\delta F}{\delta a_i}\right] + \sum_{j=1}^{m}\left[\partial, \frac{\delta F}{\delta b_j}\right]$$

$$= -\sum_{i=1}^{n}\operatorname{res}\left[A_i, A_{i+1} \ldots A_n B_1^{-1} \ldots B_m^{-1}\frac{\delta F}{\delta L} A_1 \ldots A_{i-1}\right]$$

$$+\sum_{j=1}^{m}\operatorname{res}\left[B_j, B_j^{-1} \ldots B_m^{-1}\frac{\delta F}{\delta L} A_1 \ldots A_n B_1^{-1} \ldots B_j^{-1}\right]$$

$$= -\operatorname{res}\left[L, \frac{\delta F}{\delta L}\right] \tag{39}$$

which implies

$$\sum_{i=1}^{n}\frac{\delta F}{\delta a_i} + \sum_{j=1}^{m}\frac{\delta F}{\delta b_j} = -\int^{x}\operatorname{res}\left[L, \frac{\delta F}{\delta L}\right]. \tag{40}$$

Now

$$\int \left(\sum_{i=1}^{n}\frac{\delta F}{\delta a_i} + \sum_{j=1}^{m}\frac{\delta F}{\delta b_j}\right)\left(\sum_{i=1}^{n}\frac{\delta G}{\delta a_i} + \sum_{j=1}^{m}\frac{\delta G}{\delta b_j}\right)'$$

$$= -\int \left(\sum_{i=1}^{n}\frac{\delta F}{\delta a_i} + \sum_{j=1}^{m}\frac{\delta F}{\delta b_j}\right)'\left(\sum_{i=1}^{n}\frac{\delta G}{\delta a_i} + \sum_{j=1}^{m}\frac{\delta G}{\delta b_j}\right)$$

$$= -\int \operatorname{res}\left([L, \frac{\delta F}{\delta L}]\int^{x}\operatorname{res}\left[L, \frac{\delta G}{\delta L}\right]\right)$$

$$= \int \operatorname{res}\left(\frac{\delta F}{\delta L}\left[L, \int^{x}\operatorname{res}\left[L, \frac{\delta G}{\delta L}\right]\right]\right). \tag{41}$$

□

In summary, we have shown that the second Hamiltonian structure of the cmKP hierarchy has a very simple realization. In terms of the variables $\{w_i\}$, the Lax operator K_n can be factorized as (6) and the Poisson structure (3) is mapped into a much simpler form (7). We also discuss the Poisson structure (10) under factorization of the Lax operator containing inverse linear terms. The resulting brackets (32) turns out to be simple as well. We hope that we can explore the usage of these brackets in the future.

Note added in proof. After submission of this manuscript for publication we became aware of the preprint by Liu [19] which partly overlaps our work.

We would like to thank Professor W-J Huang for inspiring discussions and Dr M-C Chang for reading the manuscript. This work was supported by the National Science Council of the People's Republic of China under grant no NSC-86-2112-M-007-020.

References

[1] Bouwknegt P and Schoutens K (ed) 1995 *W-symmetry* (Singapore: World Scientific)
[2] Dickey L A 1991 *Soliton Equations and Hamiltonian Systems* (Singapore: World Scientific)
[3] Di Francesco P, Itzykson C and Zuber J B 1991 *Commun. Math. Phys.* **140** 543
[4] Dickey L A 1993 Lecture on classical W-algebras, unpublished
[5] Kupershmidt B A and Wilson G 1981 *Inv. Math.* **62** 403
[6] Dickey L A 1983 *Commun. Math. Phys.* **87** 127
[7] Cheng Y 1995 *Commun. Math. Phys.* **171** 661
[8] Bonora L and Xiong C S 1994 *J. Math. Phys.* **35** 5781
[9] Bonora L, Liu Q P and Xiong C S 1996 *Commun. Math. Phys.* **175** 177
[10] Dickey L A 1995 *Lett. Math. Phys.* **35** 229
[11] Yu F 1993 *Lett. Math. Phys.* **29** 175
[12] Aratyn H, Nissimov E and Pacheva S 1993 *Phys. Lett.* B **314** 41
[13] Mas J and Ramos E 1995 *Phys. Lett.* B **351** 194
[14] Oevel W and Strampp W 1993 *Commun. Math. Phys.* **157** 51
[15] Shaw J C and Tu M H 1997 Miura and auto-Bäcklund transformations for the cKP and cmKP hierarchies *J. Math. Phys.* in press
[16] Liu Q P 1995 *Inverse Problems* **11** 205
[17] Huang W J, Shaw J C and Yen H C 1995 *J. Math. Phys.* **36** 2959
[18] Shaw J C and Tu M H 1997 Nonlocal extended conformal algebras associated with multi-constraint KP hierarchy and their free-field realizations *Preprint* solv-int/9708008
[19] Liu Q P 1997 The constrained MKP hierarchy and the generalized Kupershmidt–Wilson theorem *Preprint* solv-int/9707012